Examinee Info. 学生资料

Country/国 家: ________ City/市（省）: ________ School/学 校: ________

Name/姓 名: ________ Sex/性 别: ________ Instructor/辅导老师: ________

2013年第九屆“IMC國際數學競賽”(新加坡)
Ninth IMC International Mathematics Contest (Singapore),2013

Senior High School Second Year Contest Problem

Examination Time：90 min Total Point：100points Score：________

- Contestant must write down the answer of each problem in the blank, answer with erasure will not be credited!
- For Problem 17 and 18, presentation of solution on the space provided is a must, no credit will be given if only the final answer is written down in the paper!

Multiple Choice	1	2	3	4	5	6	7	8
Answer								
Fill-in the blank	9	10	11	12	13	14	15	16
Answer								

A. Multiple-Choice Problems.（Each problem worth 5 points, a total of 40 points）

1. Consider Statement *A*: $x \neq 1006$ or $y \neq 1007$ and Statement *B*: $x + y \neq 2013$.
   What is the relation of statement *A* to that of statement *B*?
   （A）Unnecessary and sufficient condition
   （B）Necessary but not sufficient condition
   （C）Necessary and Sufficient Condition
   （D）not sufficient nor necessary condition

2. Select any three distinct numbers from $\{1, 2, 3, \cdots, 20\}$ so that when these three numbers are arranged in arithmetic sequence of ascending order. How many possible arithmetic sequences are there?
   （A）90 （B）120 （C）180 （D）240

3. There are some sort of important documents locked in a safety box, there is only one key from the *n* keys that can really open the safety box. If we want to open the safety box using each of the available keys, in average, what is the number of times of trials needed in order to open the safety box?
   （A）1 （B）n （C）$\frac{n+1}{2}$ （D）$\frac{n-1}{2}$

4. If the graph of a circle, $x^2 + y^2 = n^2$ intersects the maximum and minimum points of the function $f(x) = \sqrt{3}\sin\frac{\pi x}{n}$. What is the value of the minimum positive integer *n*?
   （A）1 （B）2 （C）3 （D）4

5. If the coordinate (x_0, y_0) is the intersection point of straight line $x + y = 2k - 1$ and circle $x^2 + y^2 = k^2 + 2k - 3$, what is the real value of k such that $x_0 y_0$ will be maximum?
   （A）$2+\frac{\sqrt{2}}{2}$ （B）1 （C）$2-\frac{\sqrt{2}}{2}$ （D）-3

6. Let θ be an angle whose terminal side at second quadrant and $\sin\frac{\theta}{2} < \cos\frac{\theta}{2}$. What is the simplified value of $2^{\left|\log_2\left|\cos\frac{\theta}{2}\right|\right|}$?
   （A）$\sec(\pi-\frac{\theta}{2})$ （B）$\cos\frac{\theta}{2}$ （C）$\sec(-\frac{\theta}{2})$ （D）$-\cos(\frac{\theta}{2}-\pi)$

7. Mary randomly selects 3 digits from1 to 9 and then arranges these as a 3-digit number with the bigger digit in the higher place value. Nestor also randomly selects 3 digits from 1 to 8 and follows the above procedure to form another 3-digit number. What is the probability that the 3-digit number Mary has selected is greater than the 3-digit number which Nestor has selected?
   （A）$\frac{47}{72}$ （B）$\frac{49}{72}$ （C）$\frac{37}{56}$ （D）$\frac{2}{3}$

8. The side length of an equilateral △ABC is a unit. Fold $\overline{PQ}$ such that it is parallel to $\overline{BC}$ （where P, Q lie on $\overline{AB}$, $\overline{AC}$; respectively), so that plane PAQ⊥ plane PQC. If the length of $\overline{AB}$ is d after folding, then what is the least possible value of d?
   (A) $\frac{\sqrt{10}}{4}a$ (B) $\frac{\sqrt{5}}{4}a$ (C) $\frac{3}{4}a$ (D) $\frac{\sqrt{3}}{4}a$

B. Fill in the blank.（5 points each, a total of 40 points）

9. Find the solution set of $|x|^3 - 2x^2 - 4|x| + 3 < 0$.

10. The edge length of a cube $ABCD\text{-}A_1B_1C_1D_1$ is 1 unit such that *M*,*N* and *P* are the midpoints of $\overline{B_1C_1}$, $\overline{C_1D_1}$, and $\overline{D_1D}$; respectively. What is the area of the cross-section formed by three points *M*, *N*, and *P* of the given cube?

11. Let $A(-1,\ 0)$ and $B(1,\ 0)$ be two points on a rectangular plane, point P lies on the circle C: $(x-3)^2+(y-4)^2=4$. What is the maximum value of $|AP|^2+|BP|^2$?

12. Let real numbers x, y satisfy the system of equations $\begin{cases}(x-1)^3+2005(x-1)=-1,\\(y-1)^3+2005(y-1)=1.\end{cases}$ What is the numerical value of $x+y$?

13. Give the prime factors of $1+2^{21}+4^{21}$.

14. If the focal length of an ellipse, the length of the minor axis, length of major axis will form an appropriate order of arithmetic sequence, what is the sum of the square of all possible values of eccentricity?

15. Procedure of performing a ***bubble sort*** operation to a series $a_1, a_2, \cdots, a_n$ is as follows: First, compare the size of a1 from the first and a2 from the second terms in the given series. If $a_1>a_2$, then swap the two positions in the series (that is, exchange their value), or remain unchanged if they are the equal; then compare the size of second and third term in the series, follow the same rule, that is; if $a_2>a_3$, then swap, or else remain unchanged if they are equal; then compare the size of the third and fourth and so on, $\cdots$, according to the same rules, when reaching the last two terms, after compare the size of a_{n-1}, a_n and after the end of swapping the location, then stop. Now randomly arrange $1, 2, 3, \cdots, 2013$ as a series and perform the above described "Bubble Sort" technique to this series of 2013 terms. What is the probability that after bubble sort the number in the 10^{th} term is the number in the 5^{th} term?

16. Let $f(x)=\left|\left|\cdots\left|x^{10}-2^{2013}\right|-2^{2012}\right|-\cdots-2^2\right|-2\right|$. What is the value of $f(2013)$?

C. Problem Solving. (10 points each, a total of 20 points. Show your detailed solution on the space below each question)

17. Given: $A\subseteq\{1,2,3,\cdots\cdots,2013\}$. What is the largest possible number of pairs of elements in set A so that the absolute value of the difference of any two elements of the set is neither equal to 3 nor 4 nor 5?

8. The triplet (A, B, C) can be transformed with the following two steps:
Transformation 1: Three numbers can be rearranged arbitrarily
Transformation 2: Convert (A, B, C) into ($2B+2C-A$, B, C);
Assume that the initial state of a given triplet is (−1, 0, 1).
(a) Can we perform a finite number of steps of transformation to obtain a triplet as (2012, 2013, 2014). Explain.

(b) Can we perform a finite number of steps of transformation to obtain a triplet such as (2009, 2010, 2011). Explain.

(c) Determine all possible values of x so that there is a finite number of steps of transformation to obtain a triplet as (1, 2024, x). Explain